UDC 62-50

CONVERGENCE OF THE SOLUTION OF THE LINEAR SINGULARLY PERTURBED PROBLEM OF TIME-OPTIMAL RESPONSE

PMM Vol. 43, No. 3, 1979, pp. 466-474,
T. R. GICHEV and A. L. DONCHEV
(Sofiia)
(Received April 18, 1978)

The problem of time-optimal response of a linear control system is considered. Convergence of the solution of this problem to the solution of the problem of time-optimal response for a trunkated system is studied under specified conditions.

1. Let the behavior of the controlled system be described by the following vector differential equation:

$$x^{\cdot} = A_0 x + B_0 u, \quad x(0) = v \tag{1.1}$$
$$x \in R^n, \quad u \in \Omega \subset R^r$$

Here Ω is a compact convex polygon, and the coordinate origin O_r of the space R^r belongs to the interior of the space Ω, while A_0 and B_0 are constant matrices, $n \times n$ and $n \times r$, respectively. The set of admissible controls consists of the piecewise continuous functions $u(t)$ defined on the finite time intervals $[0, t_1]$. Any admissible control has a finite number of points of discontinuity belonging to the interval $(0, t_1)$, and is continuous from the right of these points.

The problem of time-optimal response for the system (1.1) (see [1, 2]) consists of finding an admissible control which would take it from the fixed initial state v into the coordinate origin O_n of the space R^n in a shortest possible time (problem Γ_0). Let the behavior of the controlled system with $\lambda \in (0, \Lambda)$, $\Lambda > 0$ be described by the following vector equation:

$$x^{\cdot} = A_{11}x + A_{12}y + B_1 u, \quad x(0) = v \tag{1.2}$$
$$\lambda y^{\cdot} = A_{21}x + A_{22}y + B_2 u, \quad y(0) = w; \quad y \in R^m$$

where A_{ij} and B_i are constant matrices of the corresponding dimensions. We shall also consider for this system the problem of time-optimal response, which consists of finding an admissible control taking it from the fixed initial state (v, w) to the coordinate origin O of the space R^{m+n} in a shortest possible time (problem Γ_λ).

The question of how regular perturbations affect the solutions of the problems of linear, time-optimal response was studied in [3, 4]. A problem of time-optimal response with a singular perturbation was formulated in [4] and certain asymptotic properties of its solution were discussed.

Below we shall investigate the convergence of the solution of the problem Γ_λ to the solution of Γ_0 as $\lambda \to 0$, with the matrices A_0 and B_0 defined as follows:

$$A_0 = A_{11} - A_{12}A_{22}^{-1}A_{21}, \quad B_0 = B_1 - A_{12}A_{22}^{-1}B_2 \tag{1.3}$$

We shall use, on one hand, the approach adopted in [4] in the course of investigation of the correctness of the formulation of the linear problem of time-optimal response. On the other hand, we shall utilize the mathematical apparatus developed in [6, 7] while investigating the singularly perturbed problems of optimal control with the convex performance index.

We assume that the following three conditions hold:

1°. The real parts of the eigenvalues of the matrix A_{22} are negative.

2°. The condition of generality of position (see [1, 2]) holds for the equation (1.1) with the matrices A_0 and B_0 and for the polygon Ω; the problem Γ_0 has an optimal control denoted here by $u_0(t),\ 0 \leqslant t \leqslant T_0$.

3°. $\operatorname{rank}\ [B_2 A_{22} B_2 \ldots A_{22}^{m-1} B_2] = m.$

Condition 3° was used in [5].

2. Let us prove the following two auxilliary lemmas.

Lemma 2.1. Let $u(t),\ 0 \leqslant t \leqslant T+\gamma,\ 0<\gamma<T$ be an admissible control continuous at all points $t \in (T-\gamma,\ T+\gamma)$ and let the sequences $\{\lambda_k\}_1^\infty$, $\lambda_k \in (0,\ \Lambda)$, $\lim \lambda_k = 0\ (k \to \infty)$ and $\{T_k\}_1^\infty$, $T_k > 0$, $\lim T_k = T\ (k \to \infty)$ be given. Then if $x^*(t),\ 0 \leqslant t \leqslant T$ is a solution of (1.1) corresponding to the control $u(t)$ and $(x_k(t),\ y_k(t)),\ 0 \leqslant t \leqslant T_k$ is a solution of (1.2) for $u(t)$, λ_k, then

$$\lim_{k\to\infty} x_k(T_k) = x^*(T),\quad \lim_{k\to\infty}\ \max_{t\in[0,\ \min(T_k,\ T)]} \| x_k(t) - x^*(t) \| = 0 \tag{2.1}$$

$$\lim_{k\to\infty} y_k(T_k) = -A_{22}^{-1}(A_{21}x^*(T) + B_2 u(T)) \tag{2.2}$$

Proof. Let the basic solution of the homogeneous equation

$$\xi^{\cdot} = A_{11}\xi + A_{12}\eta,\quad \lambda_k \eta^{\cdot} = A_{21}\xi + A_{22}\eta$$

be denoted by

$$\Phi^k(t) = \begin{Vmatrix} \Phi_{11}^k(t) & \Phi_{12}^k(t) \\ \Phi_{21}^k(t) & \Phi_{22}^k(t) \end{Vmatrix}$$

where $\Phi^k(0)$ is a unit matrix. From the Cauchy formula we obtain

$$y_k(T_k) = \Phi_{21}^k(T_k)v + \Phi_{22}^k(T_k)w + \int_0^{T_k} \left(\Phi_{21}^k(T_k - t)B_1 + \frac{1}{\lambda_k}\Phi_{22}^k(T_k - t)B_2 \right) u(t)\,dt. \tag{2.3}$$

By virtue of Lemma 3 of [6] the matrix $\Phi_{21}{}^k(t)$ is uniformly bounded on the segment $[0,\ T+\gamma]$, and for every $t^* \in (0,\ T+\gamma]$

$$\lim_{k\to\infty} \Phi_{21}^k(t) = -A_{22}^{-1}A_{21}\exp(A_0 t)$$

uniformly on the segment $[t^*,\ T+\gamma]$. Consequently

$$\lim_{k\to\infty} \int_0^{T_k} \Phi_{21}^k(T_k - t)B_0 u(t)\,dt = -A_{22}^{-1}A_{21}\int_0^T \exp(A_0(T-t))B_0 u(t)\,dt \tag{2.4}$$

If

$$\overline{\Phi}_{22}^{k}(t)=\begin{cases}\Phi_{22}^{k}(t), & t\in[0,\ T_k]\\ \Phi_{22}^{k}(T_k), & t\in[T_k, T+\gamma]\end{cases}$$

then from the equation

$$\frac{d}{d\tau}(\Phi_{22}^{k}(t-\tau))'=-A_{12}'(\Phi_{21}^{k}(t-\tau))'-\frac{1}{\lambda_k}A_{22}'(\Phi_{22}^{k}(t-\tau))' \tag{2.5}$$

we obtain

$$\lim_{k\to\infty}\overline{\Phi}_{22}^{k}(t)=\begin{cases}\Theta_m, & t\in(0,\ T+\gamma]\\ I_m, & t=0\end{cases} \tag{2.6}$$

where I_m is a unit matrix and Θ_m is a zero matrix in R^m, while a prime denotes a transpose. The total variation of the functions $\Phi_{22}{}^{k}(t)$ is uniformly bounded on the segment $[0,\ T+\gamma]$.

From (2.6) and Helly theorem it follows that

$$\lim_{k\to\infty}\int_0^{T_k}(A_{22}^{-1}B_2u(t))'d(\Phi_{22}^{k}(T_k-t))'=(A_{22}^{-1}B_2u(T))' \tag{2.7}$$

Let us set

$$y^*(t)=-A_{22}{}^{-1}(A_{21}x^*(t)+B_2u(t))=$$

$$-A_{22}^{-1}A_{21}(\exp(A_0t)v+\int_0^t\exp(A_0(t-\tau))B_0u(\tau)d\tau-A_{22}^{-1}A_{21}B_2u(t)$$

Then from (2.4), (2.5) and (2.7) it follows that

$$\|y_k(T_k)-y^*(T)\|\leqq\|(\Phi_{21}^{k}(T_k)+A_{22}^{-1}A_{21}\exp(A_0T))v\|+$$

$$\|\Phi_{22}^{k}(T_k)w\|+\left\|\int_0^{T_k}\Phi_{21}(T_k-t)B_0u(t)dt+\int_0^{T}A_{22}^{-1}A_{21}\exp(A_0(T-t))\times\right.$$

$$\left.B_0u(t)dt\right\|+\|A_{22}^{-1}B_2u(T)-\int_0^{T_k}(A_{22}^{-1}B_2u(t))'d(\Phi_{22}^{k}(T_k-t))')'\|$$

i.e. (2.2) holds. In the same manner we show that the sequence $\{y_k(t)\}_1^\infty$ converges almost everywhere on the segment $(0, T)$ to $y^*(t)$. But the sequence $\{y_k(t)\}_1^\infty$, $0\leqslant t\leqslant T_k$ is uniformly bounded, therefore the equation

$$x^*(T)-x_k(T_k)=\int_0^T\Phi_0(T-t)(A_{12}y^*(t)+B_1u(t))dt-$$

$$\int_0^{T_k}\Phi_0(T_k-t)(A_{21}y_k(t)+B_1u(t))dt$$

implies that $\lim x_k(T_k)=x^*(T)$ $(k\to\infty)$, where $\Phi_0(t)$ is the fundamental matrix of the equation $\xi^{\cdot}=A_{11}\xi$. The second equation of (2.1) is proved in the same manner.

Let us denote by $K(T,\lambda)$ the set of attainability, with the initial state (v, w)

and the admissible controls $u(t)$, $0 \leqslant t \leqslant T$ with $\lambda \in (0, \Lambda)$. We know that the set $K(T, \lambda)$ is convex and compact for any $\lambda \in (0, \Lambda)$ and $T > 0$. Let us denote by $K(T)$ the set of attainability for the system (1.1) with the initial state v and with admissible controls $u(t)$, $0 \leqslant t \leqslant T$.

L e m m a 2.2. For every $\varepsilon > 0$ there exists $\delta > 0$ such that when $\lambda \in (0, \delta)$, then $O \in K(T_0 + \varepsilon, \lambda)$.

P r o o f. We assume the opposite. Then a number $\varepsilon_0 > 0$ and a sequence $\{\lambda_k\}_1^\infty$, $\lambda_k \in (0, \Lambda)$, $\lim \lambda_k = 0$, $(k \to \infty)$, can be found such that $0 \notin K(T_0 + \varepsilon_0, \lambda_k)$ for $k = 1, 2, \ldots$. Then from a known theorem of convex analysis it follows that for every $k = 1, 2, \ldots$ there exists a vector (p_k, q_k), $p_k \in R^n$, $q_k \in R^m$, $\|(p_k, q_k)\| = 1$ such that the inequality

$$p_k' x + q_k' y < 0 \tag{2.8}$$

holds for all $(x, y) \in K(T_0 + \varepsilon_0, \lambda_k)$. We can assume without restricting the generality, that $\lim (p_k, q_k) = (p_0, q_0)$ $(k \to \infty)$. Let us introduce the matrix

$$M_k = \frac{1}{\lambda_k} \int_{t_k}^{T_0+\varepsilon_0} E_{22}(t) B_2 B_2' E_{22}'(t)\, dt$$

$$E_{22}(t) = \exp\left(A_{22} \frac{T_0 + \varepsilon_0 - t}{\lambda_k}\right), \quad t_k = T_0 + \varepsilon_0 - \sqrt{\lambda_k}$$

The authors show in Lemma 2 of [7] that condition 3° implies that $\lim M_k = M_0$ $(k \to \infty)$, and the matrix M_0 is nondegenerate. Let a number $\sigma > 0$ be chosen so that if $\|u\| < \sigma$, then $u \in \Omega$. The numbers $\varepsilon_1^* > 0$, $\alpha_0 > 0$ and $\beta > 0$ are chosen in such a manner, that the following inequalities hold for $\alpha \in (0, \alpha_0)$ and $\varepsilon_1 \in (0, \varepsilon_1^*)$:

$$\begin{aligned} &\varepsilon_1 \sup_k \max_{t \in [T_0, T_0+\varepsilon_0]} \|B_2' E_{22}'(t) M_k^{-1} A_{22}^{-1}\| \|\exp(A_0(T_0 + \varepsilon_0 - t)) B_0\| \max_{u \in \Omega} \|u\| < \frac{\sigma}{3} \\ &\alpha \sup_k \max_{t \in [T_0, T_0+\varepsilon_0]} \|B_2' E_{22}'(t) M_k^{-1} A_{22}^{-1} A_{21} p_0\| < \frac{\sigma}{3} \\ &\beta \sup_k \max_{t \in [T_0, T_0+\varepsilon_0]} \|B_2' E_{22}'(t) M_k^{-1} q_0\| < \frac{\sigma}{3} \end{aligned} \tag{2.9}$$

From the condition 2° it follows that a number $\alpha \in (0, \alpha_1)$ and an admissible control $\bar{u}(t)$, $T_0 \leqslant t \leqslant T_0 + \varepsilon_0$ exist, which transport the phase point from the state O_n to the state αp_0. Let the number $\varepsilon_1 \in (0, \varepsilon_1^*)$ be fixed so that the following inequality holds:

$$\alpha \|p_0\|^2 + p_0' p^* + \beta \|q_0\|^2 > 0 \tag{2.10}$$

$$p^* = - \int_{T_0+\varepsilon_0-\varepsilon_1}^{T_0+\varepsilon_0} \exp(A_0(T_0 + \varepsilon_0 - t)) B_0 \bar{u}(t)\, dt$$

Then the equation

$$\bar{u}^*(t) = \begin{cases} u_0(t), & 0 \leqslant t < T_0 \\ \bar{u}(t), & T_0 \leqslant t < T_0 + \varepsilon_0 - \varepsilon_1 \\ O_r, & T_0 + \varepsilon_0 - \varepsilon_1 \leqslant t \leqslant T_0 + \varepsilon_0 \end{cases}$$

will be admissible and, if $(\bar{x}_k^*(t), \bar{y}_k^*(t))$, $0 \leqslant t \leqslant T_0 + \varepsilon_0$ is the corresponding trajectory for λ_k, then according to Lemma 2.1 the following relations will hold:

$$\lim_{k\to\infty} \bar{x}_k^*(T_0+\varepsilon_0) = \alpha p_0 + p^* \tag{2.11}$$
$$\lim_{k\to\infty} \bar{y}_k^*(T_0+\varepsilon_0) = -A_{22}^{-1}A_{21}(\alpha p_0 + p^*)$$

From (2.9) it follows that the control

$$u_k^*(t) = \begin{cases} \bar{u}^*(t), \ 0 \leqslant t < t_k \\ \delta u_k(t) = B_2' E_{22}'(t) M_k^{-1} \ (\beta q_0 + A_{22}^{-1}A_{21}(p^* + \alpha p_0)), t_k \leqslant t \leqslant T_0 + \varepsilon_0 \end{cases}$$

is admissible for all, sufficiently large k. Let us denote by (x_k^*, y_k^*) the corresponding trajectory for λ_k.

By virtue of Lemma 3 of [6], the sequences $\{\Phi_{11}^k(t)\}_1^\infty$ and $\{\lambda_k^{-1}\Phi_{12}^k(t)\}_1^\infty$ are uniformly bounded on the segment $[0, T_0+\varepsilon_0]$. From the uniform boundedness of the sequence $\{\delta u_k(t)\}_1^\infty$ and the Cauchy formula for $t \in (t_k, T_0+\varepsilon_0]$

$$x_k^*(t) - \bar{x}_k^*(t) = \int_{t_k}^{t} \left(\Phi_{11}^k(t-\tau) B_1 + \frac{1}{\lambda_k} \Phi_{12}^k(t-\tau) B_2\right) \delta u_k(\tau)\, d\tau$$

it follows that the sequence $\{(x_k^*(t) - \bar{x}_k^*(t)\}_1^\infty$ converges uniformly to zero on the segment $[0, T_0+\varepsilon_0]$. But then the first equation of (2.11) implies that

$$\lim_{k\to\infty} x_k^*(T_0+\varepsilon_0) = \alpha p_0 + p^*$$

It can easily be shown that

$$\lim_{k\to\infty} \frac{1}{\lambda_k} \int_0^{T_0+\varepsilon_0} E_{22}(t) A_{12}(x_k^*(t) - \bar{x}_k^*(t))\, dt = 0$$

Then

$$y_k^*(T_0-\varepsilon_0) - \bar{y}_k^*(T_0+\varepsilon_0) = \frac{1}{\lambda_k}\int_{t_k}^{T_0+\varepsilon_0} E_{22}(t)(A_{12}(x_k^*(t) -$$
$$\bar{x}_k^*(t)) + B_2\delta u_k(t))\, dt = \int_0^{T_0+\varepsilon_0} E_{22}(t) A_{12}(x_k^*(t) - \bar{x}_k^*(t))\, dt +$$
$$\beta q_0 + A_{22}^{-1}A_{21}(p^* + \alpha p_0)$$

This, together with the second equation of (2.11), implies that

$$\lim_{x\to\infty} y_k^*(T_0+\varepsilon_0) = \beta q_0$$

But since $(x_k^*(T_0+\varepsilon_0), y_k^*(T_0+\varepsilon_0)) \in K(T_0+\varepsilon_0, \lambda_k)$, then from (2.8) it follows that the inequality

$$p_k' x_k^*(T_0+\varepsilon_0) + q_k' y_k^*(T_0+\varepsilon_0) < 0$$

holds. On passing to the limit, the latter inequality yields

$$\alpha\|p_0\|^2 + p_0'p^* + \beta\|q_0\|^2 \leqslant 0$$

which contradicts (2.10), and this completes the proof of the lemma.

Theorem 2.1. For every number $\varepsilon > 0$ there exists a number $\delta > 0$ such, that for $\lambda \in (0, \delta)$ the problem Γ_λ has a solution. Also, if $T(\lambda)$ is the optimal time of passage, then the inequality $|T(\lambda) - T_0| < \varepsilon$ holds.

Proof. Let the number $\varepsilon > 0$ be fixed. According to Lemma 2.2. a number $\delta_1 > 0$ exists such, that for $\lambda \in (0, \delta_1)$ the point $O \in K(T_0 + \varepsilon, \lambda)$. But then, by virtue of [8] an optimal control exists for all these values of λ and the inequality

$$T(\lambda) \leqslant T_0 + \varepsilon \tag{2.12}$$

holds.

Let us assume that a sequence $\{\lambda_k\}_1^\infty$, $\lim \lambda_k = 0$, exists such, that $\lim T(\lambda_k) = T^* < T_0$ $(k \to \infty)$. Let $u_k(t)$, $0 \leqslant t \leqslant T(\lambda_k)$ be the optimal control for λ_k. The admissible control

$$u_k^*(t) = \begin{cases} u_k(t), & 0 \leqslant t < T(\lambda_k) \\ O_r, & T(\lambda_k) \leqslant t \leqslant T^* + {}^1/_2(T_0 - T^*) \end{cases}$$

transports the phase point for each k, according to (1.2), from the state (v, w) to the state O. We can assume without loss of generality that the sequence $\{u_k^*\}_1^\infty$ converges weakly to $\bar{u}$ in $L_2^{(r)}(0, T^* + {}^1/_2(T_0 - T^*))$. Then from Lemma 4 of [6] it follows that the corresponding trajectory x_k^* converges pointwise to the solution $\bar{x}$ of (1.1), the latter corresponding to the control $\bar{u}$, and the relation $\bar{x}(T^* + {}^1/_2(T_0 - T^*)) = 0_n$, holds, i.e. $O_n \in K(T^* + {}^1/_2(T_0 - T^*))$. The contradiction thus reached shows that a number $\delta \in (0, \delta_1)$ exists such that the inequality $T(\lambda) \geqslant T_0 - \varepsilon$, holds for $\lambda \in (0, \delta)$ and this, together with (2.12), proves the second part of the theorem.

3. Let us now denote by $D(\lambda)$ the set of optimal controls for the problem Γ_λ. We turn our attention to the problem of convergence of the optimal controls and trajectories of the problem Γ_λ. Since we shall employ the Pontriagin maximum principle [1, 2], certain properties of the set of attainability and of the conjugated systems will be useful. These properties shall be proved in the lemmas that follow.

Lemma 3.1. Let the sequence $\{\lambda_k\}_1^\infty$, $\lambda_k \in (0, \Lambda)$, $\lim \lambda_k = 0$ $(k \to \infty)$ and the sequence $\{(p_k, q_k)\}_1^\infty$, $p_k \in R^n$, $q_k \in R^m$ of the unit external normals (p_k, q_k) to the sets $K(T(\lambda_k), \lambda_k)$ at the point O be both given such, that $\lim (p_k, q_k) = (p_0, q_0)$. Then $q_0 = O_m$.

Proof. Assume that $\|q_0\| \neq 0$. Let $t_k = T(\lambda_k) - \sqrt{\lambda_k}$ and choose the numbers $\varepsilon_1 > 0$ and $\beta > 0$ so that the following relations hold:

$$p_0' p^* + \beta \|q_0\|^2 > 0 \tag{3.1}$$

$$B_2' \exp\left(A_{22}' \frac{T_0 + \varepsilon_1 - t}{\lambda_k}\right) M_k^{-1} (A_{22}^{-1} A_{21} p^* + \beta q_0) \in \Omega, \quad t_k \leqslant t \leqslant T(\lambda_k)$$

$k = 1, 2, \ldots$

$$p^* = -\int_{T_0 - \varepsilon_1}^{T_0} \exp(A_0(T_0 - \tau)) B_0 u_0(\tau)\, d\tau$$

Let also

$$\bar{u}(t) = \begin{cases} u_0(t), & 0 \leqslant t < T_0 - \varepsilon_1 \\ O_r, & T_0 - \varepsilon_1 \leqslant t \leqslant T_0 + \varepsilon_1 \end{cases}$$

and $(\bar{x}_k, \bar{y}_k)$ be the trajectory for λ_k corresponding to the control $\bar{u}$. Then from Lemma 2.1 and Theorem 2.1 it follows that

$$\lim_{k\to\infty} (\bar{x}_k(T(\lambda_k)), \bar{y}_k(T(\lambda_k))) = (p^*, -A_{22}^{-1}A_{21}p^*)$$

The admissible control

$$u_k^*(t) = \begin{cases} \bar{u}(t), & 0 \leqslant t < t_k \\ B_2' \exp\left(A_{22}' \dfrac{T_0 + \varepsilon_1 - t_k}{\lambda_k}\right) M_k^{-1}(\beta q_0 + A_{22}^{-1}A_{21}p^*), & t_k \leqslant t \leqslant T(\lambda_k) \end{cases}$$

has the corresponding trajectory (x_k^*, y_k^*) at λ_k. Repeating a part of the proof of Lemma 2.2, we obtain

$$\lim_{k\to\infty} (x_k^*(T(\lambda_k)), \quad y_k^*(T(\lambda_k))) = (p^*, \beta q_0)$$

But since $(x_k^*(T(\lambda_k)), y_k^*(T(\lambda_k))) \in K(T(\lambda_k), \lambda_k)$, then the definition of the vectors (p_k, q_k), $k = 1, 2, \ldots$ implies that

$$p_k' x_k^*(T(\lambda_k)) + q_k' y_k^*(T(\lambda_k)) \leqslant 0$$

A passage to the limit now yields the inequality

$$p_0' p^* + \beta \| q_0 \|^2 \leqslant 0$$

which contradicts the inequality (3.1), and this completes the proof of the lemma.

Lemma 3.2. In the assumptions of Lemma 3.1 the vector p_0 represents the outward normal to the set $K(T_0)$ at the point O_n.

Proof. Assume the opposite. Then we can find a point $v^* \in K(T_0)$ and a corresponding control $u^*(t)$, $0 \leqslant t \leqslant T_0$ such, that the following inequality holds:

$$p_0' v^* > 0 \tag{3.2}$$

Let ε_1 be any posititive number and let define the admissible control $\bar{u}$ as follows:

$$\bar{u}(t) = \begin{cases} u^*(t), & 0 \leqslant t < T_0 \\ u^*(T_0), & T_0 \leqslant t \leqslant T_0 + \varepsilon_1 \end{cases}$$

If $(\bar{x}_k, \bar{y}_k)$ is the corresponding trajectory for λ_k, then by virtue of Lemma 2.1

$$\lim_{k\to\infty} (\bar{x}_k(T(\lambda_k)), \bar{y}_k(T(\lambda_k))) = (v^*, -A_{22}^{-1}(A_{21}v^* + B_2 u^*(T_0)))$$

On the other hand, $(\bar{x}_k(T(\lambda_k)), \bar{y}_k(T(\lambda_k))) \in K(T(\lambda_k), \lambda_k)$ and consequently $p_k' \bar{x}_k (T(\lambda_k)) + q_k' \bar{y}_k (T(\lambda_k)) \leqslant 0$. Passing now to the limit and applying Lemma 3.1 we arrive at a contradiction, and this completes the proof of the lemma.

Let p_0 be an an outward normal to the set $K(T_0)$ at the point O_n, and let the function φ be a solution of the equation

$$\dot{\varphi} = -A_0'\varphi, \quad \varphi(T_0) = p_0 \tag{3.3}$$

Then for every $t \in [0, T_0]$ the optimal control u_0 (see [1,2]) satisfies the maximum condition

$$\varphi'(t) B_0 u_0(t) = \max_{u\in\Omega} \varphi'(t) B_0 u \tag{3.4}$$

Similarly, if (p, q), $p \in R^n$, $q \in R^m$ is an outward unit normal to the set $K(T(\lambda), \lambda)$, $\lambda \in (0, \Lambda)$ at the point O and the function (φ, ψ) is a solution of the system

$$\begin{aligned} \varphi^{\cdot} &= -A_{11}'\varphi - A_{21}'\psi, \quad \varphi(T(\lambda)) = p \\ \lambda\psi^{\cdot} &= -A_{12}'\varphi - A_{22}'\psi, \quad \psi(T(\lambda)) = q / \lambda \end{aligned} \tag{3.5}$$

then any optimal control $u \in D(\lambda)$ will satisfy, for every $t \in [0, T, (\lambda))$ the maximum condition

$$(\varphi'(t) B_1 + \psi'(t) B_2) u(t) = \max_{u \in \Omega} (\varphi'(t) B_1 + \psi'(t) B_2) u \tag{3.6}$$

Proof of the lemma given below follows from Lemma 1 (ii) of [7].

L e m m a 3 . 3. Let the sequences $\{\lambda_k\}_1^\infty$ and $\{(p_k, q_k)\}_1^\infty$ which satisfy the assumptions of Lemma 3.1 be given, let (φ_k, ψ_k), $k = 1, 2, \ldots$ be the solution of the equation (3.5) with the final term $(p_k, q_k / \lambda_k)$ and φ be the solution of (3.3) with the final term p_0. Then for any $T^* \in (0, T_0)$ we have

$$\lim_{k\to\infty} \max_{t\in[0, T^*]} (\| \varphi_k(t) - \varphi(t) \| + \| \psi_k(t) + A_{22}^{-1} A_{21} \varphi(t) \|) = 0 \tag{3.7}$$

T h e o r e m 3 . 1. For every number $\varepsilon > 0$ there exists a number $\delta > 0$ such, that if $\lambda \in (0, \delta)$ and $u \in D(\lambda)$, then a finite number of open intervals Δ_i, $\operatorname{mes}(\bigcup_i \Delta_i) < \varepsilon$ can be found for which $u(t) = u_0(t)$ when $t \in [0, T_0] \setminus \bigcup_i \Delta_i$.

P r o o f. Assume the opposite. Then a number $\varepsilon_0 > 0$, a sequence $\{\lambda_k\}_1^\infty$, $\lambda_k \in (0, \Lambda)$, $\lim \lambda_k = 0$ $(k \to \infty)$ and a control $u_k \in D(\lambda_k)$ can be found for which the statement of the theorem will be false. Let (p_k, q_k) be a unit outward normal to the set $K(T_k, \lambda_k)$ at the point O where $T(\lambda_k) = T_k$. We can assume without loss of generality that the sequence $\{(p_k, q_k)\}_1^\infty$ converges to (p_0, q_0) and, in accordance with Lemma 3.1, $\| q_0 \| = 0$ while Lemma 3.2. implies that the vector p_0 is an outward normal to the set $K(T_0)$ at the point O_n. Let (φ_k, ψ_k) be a solution of (3.5) with the final term $(p_k, q_k / \lambda_k)$ and φ a solution of (3.3) with the final term p_0.

Let $0 \leqslant \tau_1 < \tau_2 < \ldots < \tau_{l-1} < T_0$ denote all instants of time at which the control $u_0(t)$ cannot be uniquely determined from the maximum condition (3.4). The set $\{\tau_i, i = 1, \ldots, (l-1); T_0\}$ can be overlapped by the open, nonintersecting intervals Δ_i, $i = 1, \ldots, l$ such that $\operatorname{mes}(\bigcup_i \Delta_i) < \varepsilon_0 / 2$. Since by virtue of Theorem 2.1 $\lim T_k = T_0 (k \to \infty)$, we can assume that $T_k \notin [0, T_0] \setminus \bigcup_i \Delta_i$. If $T^* \in \Delta_l$ and $T^* < T_k$, then from Lemma 3.3. it follows that the relation (3.7) holds.

Now, in accordance with the assumptions made above a sequence of points $\{t_k\}_1^\infty$, $t_k \in [0, T_0] \setminus \bigcup_i \Delta_i$, $\lim t_k = t^*$ $(k \to \infty)$ can be found such, that

$$u_k(t_k) \neq u_0(t_k), \quad u_k(t_k) = \bar{u}, u_0(t_k) = u^*, \ \bar{u} \neq u^*$$

But from (3.6) it follows that

$$(\varphi_k'(t_k) B_1 + \psi_k'(t_k) B_2) \bar{u} \geqslant (\varphi_k'(t_k) B_1 + \psi_k'(t_k) B_2) u^*$$

from which, passing to the limit and utilizing (3.7), we obtain

$$\varphi'(t^*)B_0\bar{u} \geqslant \varphi'(t^*)B_1u^*$$

i.e. $\bar{u} = u^*$ which is a contradiction and hence proves the theorem.

Let $x_0(t)$, $0 \leqslant t \leqslant T_0$ be the optimal trajectory in the problem Γ_0, and $y_0(t) = -A_{22}^{-1}(A_{21}x_0(t) + B_2u_0(t))$.

T h e o r e m 3 . 2. For every $\varepsilon > 0$ there exists $\delta > 0$ such that if $\lambda \in (0, \delta)$ and $u \in D(\lambda)$, then a finite number of open intervals Δ_i, $i = 1, \ldots, l$, $\operatorname{mes}(\bigcup_i \Delta_i) < \varepsilon$ can be found for which the following relations hold:

$$\max_{t \in [0, \min(T(\lambda), T_0)]} \| x(t, \lambda) - x_0(t) \| < \varepsilon \tag{3.8}$$

$$\max_{t \in [0, T_0] \setminus \bigcup_i \Delta_i} \| y(t, \lambda) - y_0(t) \| < \varepsilon$$

where the trajectory $(x(t, \lambda), y(t, \lambda))$ corresponds to the control u and the value of the parameter λ.

We prove the theorem again by assuming the opposite. Let a number $\varepsilon_0 > 0$, a sequence $\{\lambda_k\}_1^\infty$, $\lambda_k \in (0, \Lambda)$, $\lim \lambda_k = 0$ $(k \to \infty)$ and controls $u_k \in D(\lambda_k)$ exist, for which at least one of the inequalities (3.8) does not hold. Let the sequences $\{\lambda_k\}_1^\infty$ and $\{u_k\}_1^\infty$ possess all the properties of the analogous sequences in the proof of Theorem 3.1.

If τ_i, $i = 1, \ldots, l-1$, $\tau_i < T_0$ are the points at which the optimal control $u_0(t)$ satisfying the maximum relation (3.4) is defined nonuniquely and $\tau_l = T_0$, then according to Theorem 3.1 there exists a sequence $\{\Delta_i^k, i = 1, \ldots, l\}_1^\infty$ of finite covers of the points τ_i such that $\lim \operatorname{mes}(\bigcup_i \Delta_i) = 0$ $(k \to \infty)$ and $u_k(t) = u_0(t)$ for $t \in [0, T_0] \setminus \bigcup_i \Delta_i^k$. Then, as in Lemma 2.1, we can prove that the sequence $\{y(t, \lambda_k)\}_1^\infty$ is bounded, converges almost everywhere in the interval $(0, T_0)$ to $y_0(t)$, and

$$\lim_{k \to \infty} \max_{t \in [0, \min(T(\lambda_k), T_0)]} \| x(t, \lambda_k) - x_0(t) \| = 0 \tag{3.9}$$

If Lemma 1 of [6] is applied at each of the segments $[\tau_i + \varepsilon_0/(8l), \tau_{i+1} - \varepsilon_0/(8l)]$, $\tau_{i+1} - \tau_i < \varepsilon_0/(4l)$, $i = 1, \ldots, l-1$, then we find that

$$\lim_{k \to \infty} \max_{t \in [0, T_0] \setminus \bigcup_i (\tau_i - \varepsilon_0/(8l), \tau_i + \varepsilon_0/(8l))} \| y(t, \lambda_k) - y_0(t) \| = 0 \tag{3.10}$$

But the relations (3.9) and (3.10) contradict the assumption made earlier that for every k at least one of the inequalities (3.8) does not hold, and this proves the theorem.

N o t e. When the problem of time-optimal response consists of finding an admissible control carrying the state of the system from the point (v, w) to the coordinate origin O_n of the space R^n in the shortest possible time, then results analogous to Theorems 2.1, 3.1 and 3.2 can be obtained with the assumption 3° omitted.

REFERENCES

1. G a m k r e l i d z e, R. V. On the theory of optimal processes in linear systems. Dokl. Akad. Nauk SSSR, Vol. 116, No. 1, 1957.

2. P o n t r i a g i n, L. S., B o l t i a n s k i i, V. G., G a m k r e l i d z e, R. V. and M i s h c h e n k o, E. F. Mathematical Theory of Optimal Processes. (English translation), Pergamon Press, Book No. 10176, 1964.

3. K i r i l l o v a, F. M. On the correctness of formulation of a problem of optimal control. Izv. vuzov. Matem. No. 4, 1958.

4. G i c h e v, T. β-continuity of the optimal control as a function of the initial state. Izv. na Matematicheskiia institut, Vol. 14, 1973.

5. K o k o t o v i č, P. V., H a d d a d, A. H. Controllability and time-optimal control of systems with slow and fast modes. IEEE Trans. Autom. Control., Vol. 20, No. 1, 1975.

6. G i č e v, T. R., D o n t c h e v, A. L. Linear optimal control system with singular perturbation and convex performance index. Serdica, Vol. 4, 1978.

7. G i c h e v, T. R. and D o n c h e v, A. L. Convergence of the solution of the problem of optimal control with free right end, in small parameter accompanying the derivative. Godishnik na Sofia Universitet, Vol. 68, 1977.

8. H a l k i n, H. A. New existence theorem in the class of piecewise continuous control functions. In: Control theory and the calculus of variations. New York—London, Acad. Press, 1969.

Translated by L. K.